Quantum Corrections to Spinning String in AdS_5 x S^5

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Summary

- Review of String / Gauge theory duality (**AdS/CFT**)

- Folded Spinning string solution
  (Gubser, Klebanov, Polyakov, 02)

- Short spinning string, Quantum corrections
  (A. Tseytlin, A. T, 08, M. Beccaria, A. T. , to appear)

- Long spinning string, Quantum Corrections
  (M. Beccaria, V. Forini, A. Tseytlin, A. T., to appear)

- Conclusions

AdS/CFT correspondence (Maldacena)

Gives a precise example of the relation between strings and gauge theory.

Gauge theory	String theory
$\mathcal{N} = 4$ SYM SU(N) on R^4	IIB on $AdS_5 x S^5$
A_μ , Φ^i, Ψ^a	radius R
Operators w/ conf. dim. Δ	String states w/ $E = \frac{\Delta}{R}$

$$g_s = g_{YM}^2 ; \qquad R/l_s = (g_{YM}^2 N)^{1/4}$$

$N \to \infty, \lambda = g_{YM}^2 N$ fixed ➡

λ large → string th.
λ small → field th.

Folded spinning string
(Gubser, Klebanov, Polyakov)

$$ds^2 = -\cosh^2\rho\; dt^2 + d\rho^2 + \sinh^2\rho\; d\phi^2$$

$$t = \kappa\tau, \quad \phi = w\tau, \quad \rho = \rho(\sigma)$$

Equation for ρ

$$\rho' = \pm\kappa\sqrt{1 - \eta\sinh^2\rho} \qquad \coth^2\rho_0 = \frac{w^2}{\kappa^2} \equiv 1 + \eta$$

Complicated classical solution

$$\sinh\rho = \frac{1}{\sqrt{\eta}}\,\mathrm{sn}\left[\kappa\sqrt{\eta}\,\sigma, -\frac{1}{\eta}\right]$$

Dual to minimal twist gauge theory operator $\mathrm{tr}(\Phi \mathrm{D}_+^S \Phi)$

Periodicity condition implies

$$\kappa = \frac{1}{\sqrt{\eta}}\ {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; -\frac{1}{\eta})$$

Energy and spin $E = \sqrt{\lambda}\,\mathcal{E} \quad S = \sqrt{\lambda}\,\mathcal{S}$

$$\mathcal{E} = \frac{1}{\sqrt{\eta}}\ {}_2F_1(-\frac{1}{2}, \frac{1}{2}; 1; -\frac{1}{\eta}) \qquad \mathcal{S} = \frac{\sqrt{1+\eta}}{2\eta\sqrt{\eta}}\ {}_2F_1(\frac{1}{2}, \frac{3}{2}; 2; -\frac{1}{\eta})$$

Cannot obtain exactly $\mathcal{E} = \mathcal{E}(\mathcal{S})$

Perturbatively in large $\mathcal{S}$

$$E - S = \frac{\sqrt{\lambda}}{\pi} \ln S + \ldots$$

$\ln S$ scaling obtained also on the gauge theory side

Difficult to quantize string on $AdS_5 \times S^5$

solution:

construct various classical solutions at quantize them semi-classically

starting action for string in $AdS_5 \times S^5$
(Metsaev, Tseytlin, 98)

$$S = T \int d^2\sigma \left[G_{mn}(x)\partial x^m \partial x^n + \bar{\theta}(D + F_5)\theta\partial x \; + \bar{\theta}\theta\bar{\theta}\theta\partial x\partial x + \ldots \right]$$

-- complicated solution – hard to quantize semi-classically even at 1-loop

-- this is the case for folded string solution

-- possible to quantize in different limits.

Short spinning string -- Quantum corrections

Folded string solution in flat space

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2$$

Solution is

$$t = \epsilon\tau \ , \qquad \rho = \epsilon \sin\sigma \ , \qquad \phi = \tau$$

string tension like in AdS

$$T = \frac{1}{2\pi\alpha'} \equiv \frac{\sqrt{\lambda}}{2\pi}$$

Classical Energy and Spin satisfy usual flat-space Regge relation

$$E_0 = \epsilon\sqrt{\lambda} \qquad\qquad S = \frac{\epsilon^2}{2}\sqrt{\lambda}$$

$$E_0(S,\lambda) = \lambda^{1/4}\sqrt{2S}$$

This is exact in flat space

Folded string solution in AdS

$$0 < \rho < \rho_{\max} \qquad \coth \rho_{\max} = \frac{w}{\kappa} \equiv \sqrt{1 + \frac{1}{\epsilon^2}}$$

$$\sinh \rho = \epsilon \, \mathrm{sn}(\kappa \epsilon^{-1} \sigma, \; -\epsilon^2)$$

ϵ measures the length of the string

We expand in small ϵ

$$\rho_{\max} = \epsilon - \frac{1}{6}\epsilon^3 + O(\epsilon^5)$$

$$\epsilon = \sqrt{2\mathcal{S}} - \frac{1}{4\sqrt{2}}\mathcal{S}^{3/2} + \ldots$$

Short string limit corresponds to small semi-classical spin $\mathcal{S} \ll 1$

Classical energy

$$E_0(S, \lambda) = \lambda^{1/4}\sqrt{2S} + \frac{3}{4\sqrt{2}}\lambda^{-1/4}S^{3/2} + O(S^{5/2})$$

This small spin expansion is an example of a near flat space expansion: the leading-order in solution ϵ can be identified with the folded spinning string solution in the flat space

Quantum corrections

$\left(\frac{1}{\sqrt{\lambda}}\right)$ Corrections respect the structure at classical level

Semiclassical quantization

$$\lambda \gg 1, \quad \frac{S}{\sqrt{\lambda}} = \text{fixed} \ll 1$$

Energy has the following structure

$$E(S,\lambda) = \lambda^{1/4}\sqrt{2S}\,\left[h_0(\lambda) + h_1(\lambda)S + h_2(\lambda)S^2 + \dots\right]$$

$$h_n = \frac{1}{(\sqrt{\lambda})^n}\left(a_{n0} + \frac{a_{n1}}{\sqrt{\lambda}} + \frac{a_{n2}}{(\sqrt{\lambda})^2} + \dots\right)$$

Classical string

$$a_{00} = 1 \ , \qquad a_{10} = \frac{3}{8} \ , \qquad a_{20} = -\frac{21}{128} \ , \ldots$$

1-loop string computation gives

$$a_{01} = 1 \ , \qquad a_{11} = \frac{41}{64} - \tfrac{1}{2}\zeta(3) \approx 0.039$$

UV finitness of superstring implies

$$h_0(\lambda) = 1$$

Gauge theory

Corresponding operator in SL(2) sector
low twist operator $\mathrm{tr}(\Phi D_+^S \Phi)$ with S ~ 1

anomalous dimension scale as

(A.V. Kotikov, L.N. Lipatov, A.I. Onishchenko, V.N. Velizhanin)

$$\Delta(S,\lambda) = q_1(\lambda)S + q_2(\lambda)S^2 + O(S^3)$$

$$q_1(\lambda) = 1 + d_{01}\lambda + d_{02}\lambda^2 + \ldots$$

$$q_2(\lambda) = d_{21}\lambda + d_{22}\lambda^2 + \ldots$$

$$\lambda \ll 1, \quad S = \text{fixed}$$

formally expanded in small S limit

cannot directly continue string expansion to small S and small λ

To relate the ``small spin'' string theory and gauge theory expansions one would need to re-sum the series in both arguments (λ, S)
sum up the weak-coupling expansion and then re-expand the result first in large λ for fixed $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$
and then in small $\mathcal{S}$

1-loop correction at strong coupling – some details

Work in conformal gauge with flat 2d metric
expand the $AdS_5 \times S^5$ superstring action near solution at quadratic order in fluctuations for bosons and fermions

$$\begin{aligned}\tilde{L}_B &= -\partial_a \tilde{t}\partial^a \tilde{t} - \mu_t^2 \tilde{t}^2 + \partial_a \tilde{\phi}\partial^a \tilde{\phi} + \mu_\phi^2 \tilde{\phi}^2 \\ &+ 4\tilde{\rho}(\kappa \sinh\rho\ \partial_0 \tilde{t} - w \cosh\rho\ \partial_0 \tilde{\phi}) + \partial_a \tilde{\rho}\partial^a \tilde{\rho} + \mu_\rho^2 \tilde{\rho}^2 \\ &+ \partial_a \beta_u \partial^a \beta_u + \mu_\beta^2 \beta_u^2 + \partial_a \varphi \partial^a \varphi + \partial_a \chi_s \partial^a \chi_s \ ,\end{aligned}$$

$$\mu_\phi^2 = 2\rho'^2 - w^2, \qquad \mu_\rho^2 = 2\rho'^2 - w^2 - \kappa^2, \qquad \mu_\beta^2 = 2\rho'^2$$

β_u $(u = 1, 2)$ AdS_5 fluctuations transverse to AdS_3

φ, χ_s $(s = 1, 2, 3, 4)$ fluctuations in S^5

The fermionic part of the quadratic fluctuation Lagrangian -- 4+4 2d Majorana fermions with σ -dependent mass

$$\tilde{L}_F = 2i(\bar{\Psi}\gamma^a \partial_a \Psi - \mu_F \bar{\Psi}\Gamma_{234}\Psi) \ , \qquad \mu_F^2 = \rho'^2$$

Expanding coefficients in small ϵ

$$\mu_t^2 = \epsilon^2 \cos 2\sigma + \ldots, \qquad \mu_\phi^2 = -1 + (\cos 2\sigma + \frac{1}{2})\epsilon^2 + \ldots,$$

$$\mu_\rho^2 = -1 + (\cos 2\sigma - \frac{1}{2})\epsilon^2 + \ldots, \qquad \mu_\beta^2 = 2\mu_F^2 = 2\epsilon^2 \cos^2\sigma + \ldots$$

Fluctuation Lagrangian is σ dependent, not easy to compute spectrum

1-loop correction to string energy

$$E_1 = \frac{\Gamma_1}{\kappa \mathcal{T}} \qquad \mathcal{T} \equiv \int d\tau \to \infty$$

Fluctuation Lagrangian does not depend on time

$$\det[-\partial_1^2 - \partial_0^2 + 2\epsilon^2 \cos^2\sigma] = \mathcal{T} \int \frac{d\omega}{2\pi} \det[-\partial_1^2 + \omega^2 + 2\epsilon^2 \cos^2\sigma]$$

We can now use perturbation theory in ϵ^2

$$\ln \frac{\det[A+\epsilon^2 B]}{\det A} = \epsilon^2 Tr[A^{-1}B] + O(\epsilon^4)$$

$$Z = \frac{\det^{\frac{8}{2}}[-\partial_0^2 - \partial_1^2 + \epsilon^2\cos^2\sigma]\ \det^{\frac{2}{2}}[-\partial_0^2 - \partial_1^2]}{\det^{\frac{2}{2}}[-\partial_0^2 - \partial_1^2 + 2\epsilon^2\cos^2\sigma]\ \det^{\frac{5}{2}}[-\partial_0^2 - \partial_1^2]\ \det^{\frac{1}{2}} Q}$$

Example: for decoupled bosons

$$\ln \frac{\det[-\partial_1^2 + \omega^2 + 2\epsilon^2\cos^2\sigma]}{\det[-\partial_1^2 + \omega^2]} \approx \epsilon^2 \sum_n \frac{2}{n^2+\omega^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos^2\sigma$$

$$= \epsilon^2 \sum_n \frac{1}{n^2+\omega^2}$$

Q is 3 x 3 matrix coupled fluctuation operator

Leading 1-loop ϵ^2 correction to energy vanishes
Expected energy is like in flat space. It should be true to all loops.

Higher order in expansion to get first non-zero coeff. ϵ^4.

$$\ln \frac{\det[A + \epsilon^2 B + \epsilon^4 C]}{\det A}$$

$$= \epsilon^2 \mathrm{Tr}[A^{-1}B] - \frac{\epsilon^4}{2}\mathrm{Tr}[A^{-1}BA^{-1}B] + \epsilon^4 \mathrm{Tr}[A^{-1}C]$$

A is massless propagator; B,C are σ -dependent insertions . Technically more involved.
The result is:

$\Gamma_1(\epsilon^4)$

$$= -\frac{\mathcal{T}\epsilon^4}{4\pi}\int_{-\infty}^{\infty} d\omega\Big\{\sum_n\Big[-\frac{7}{8}\frac{1}{n^2+w^2}-\frac{1}{32}\frac{1-8i\omega}{n^2+(\omega+i)^2}-\frac{1}{32}\frac{1+8i\omega}{n^2+(\omega-i)^2}\Big]$$

$$+ \frac{1}{2}\sum_n\Big[-\frac{\omega^2}{[n^2+(\omega+i)^2]^2}-\frac{\omega^2}{[n^2+(\omega-i)^2]^2}$$

$$+ \frac{1}{4}\frac{1}{n^2+w^2}\left(\frac{1}{(n-2)^2+\omega^2}+\frac{1}{(n+2)^2+\omega^2}\right)+\frac{1}{2}\frac{1}{[n^2+(\omega+i)^2][n^2+(\omega-i)^2]}$$

$$+ \omega^2\left(\frac{1}{(n+1)^2+\omega^2}+\frac{1}{(n-1)^2+\omega^2}\right)\left(\frac{1}{n^2+(\omega+i)^2}+\frac{1}{n^2+(\omega-i)^2}\right)$$

$$+ \frac{(1+\frac{i\omega}{2})^2}{4}\frac{1}{n^2+(\omega-i)^2}\left(\frac{1}{(n-2)^2+(\omega-i)^2}+\frac{1}{(n+2)^2+(\omega-i)^2}\right)$$

$$+ \frac{(1-\frac{i\omega}{2})^2}{4}\frac{1}{n^2+(\omega+i)^2}\left(\frac{1}{(n-2)^2+(\omega+i)^2}+\frac{1}{(n+2)^2+(\omega+i)^2}\right)\Big]\Big\}$$

Remarkable both sum and then the integral can be computed exactly

The summation gives

$$\sum_{n=3}^{\infty} S_n = \frac{\pi^2(\omega^2+1)\mathrm{csch}^2\pi\omega}{2\omega^2} + \frac{\pi(5\omega^2+4)\coth\pi\omega}{8\omega^3(\omega^2+1)} - \frac{53}{48(\omega^2+1)} - \frac{27}{32(\omega^2+4)}$$

$$+ \frac{3}{16(\omega^2+9)} + \frac{19}{96(\omega^2+16)} - \frac{5}{8\omega^2} - \frac{1}{4(\omega^2+1)^2} + \frac{6}{(\omega^2+4)^2} - \frac{1}{\omega^4}$$

1-loop correction to energy

$$E_1 = \frac{1}{\sqrt{2}}\left[\frac{41}{32} - \zeta(3)\right] \mathcal{S}^{3/2} + O(\mathcal{S}^{5/2})$$

$\zeta(3)$ also in dimensions of short operators at weak coupling

Generalization to non-zero J in S^5

String spinning in AdS, and around a big circle in S^5

Important for relation to SL(2) sector operators

$$\mathrm{tr}(D_+^S \Phi^J)$$

J interpreted as the length of the corresponding spin chain

Expanding in short string limit $\epsilon \ll 1$ two possible cases

- if $\nu = \mathcal{J} \gg 1$ fast short string, BMN like limit

$$\mathcal{E}_0 = \nu + \mathcal{S} + \frac{\mathcal{S}}{2\nu^2} + \ldots \,, \qquad \nu \gg 1, \quad \frac{\mathcal{S}}{\nu} \ll 1$$

- if $\nu \ll \sqrt{\mathcal{S}} \ll 1$ slow short string limit

Classical energy has near flat-space expansion

$$\mathcal{E}_0 = \sqrt{2\mathcal{S}}\ \left(1 + \frac{\nu^2}{4\mathcal{S}} + ...\right) + \frac{3}{4\sqrt{2}}\mathcal{S}^{3/2}\left(1 + \frac{5\nu^2}{12\mathcal{S}} + ...\right) + ...$$

1-loop computation in the second case

Result:

$$\begin{aligned} E &= \lambda^{\frac{1}{4}}\sqrt{2S}\Big[1 + \frac{J^2}{4\sqrt{\lambda}S}(1 + 0 + ...) - \frac{J^4}{32\lambda S^2}(1 + 0 + ...) + O(J^6)\Big] \\ &+ \frac{3}{4\sqrt{2}}\lambda^{-\frac{1}{4}}S^{\frac{3}{2}}\Big[\Big(1 + \frac{4}{3\sqrt{\lambda}}(\frac{41}{32} - \zeta(3)) + ...\Big) + \frac{J^2}{\sqrt{\lambda}S}\Big(\frac{5}{12} + \frac{1}{3\sqrt{\lambda}} + ...\Big) \\ &- \frac{J^4}{\lambda S^2}\Big(\frac{7}{96} + \frac{1}{12\sqrt{\lambda}} + ...\Big) + O(J^6)\Big] + O(S^{\frac{5}{2}}) \end{aligned}$$

Computed 1-loop correction to order $S^{\frac{5}{2}}$
The result contains rational numbers,

$\zeta(3)$ and $\zeta(5)$

Higher order in S, more zeta functions appear at J=0

Interesting to compute two-loop string corrections but hard, and, of course, to sum up the series

Understand strong coupling limit of anomalous dimension Δ for short operators -- finite S

Beyond asymptotic BA

Long spinning string -- Quantum Corrections

Start with spinning string solution

$$\sinh\rho = \frac{1}{\sqrt{\eta}}\,\mathrm{sn}\left[\kappa\sqrt{\eta}\,\sigma, -\frac{1}{\eta}\right]\,, \qquad 0 \le \sigma \le \frac{\pi}{2}$$

Maximum length ρ_0

$$\coth^2\rho_0 = \frac{w^2}{\kappa^2} \equiv 1 + \eta$$

Small η expansion

$\eta \to 0$ solution is $\rho = \kappa_0 \sigma$ $\qquad \kappa_0 \equiv \frac{1}{\pi} \ln \frac{16}{\eta}$

String touches the boundary of AdS $\rho_0 = \infty$

At leading order this leads to the energy E – S ~ Log S

Here we want to go to next orders in large S

Solution can be expanded as

$$\sinh\rho = \sinh(\kappa_0\sigma) - \frac{\eta}{8}\left[\sinh(2\kappa_0\sigma) - \tfrac{4}{\pi}\sigma\right]\cosh(\kappa_0\sigma) + \mathcal{O}(\eta^2)$$

Energy and spin expansion

$$\mathcal{E} = \frac{2}{\pi\eta} + \frac{\pi\kappa_0 + 1}{2\pi} - \frac{\eta}{32\pi}(2\pi\kappa_0 - 3) + \mathcal{O}(\eta^2)$$

$$\mathcal{S} = \frac{2}{\pi\eta} - \frac{\pi\kappa_0 - 3}{2\pi} - \frac{\eta}{32\pi}(2\pi\kappa_0 + 13) + \mathcal{O}(\eta^2)$$

Next to leading order string does not touch the boundary

Classical energy is given by

$$E = \sqrt{\lambda}\, \mathcal{E}(\mathcal{S}) \ , \qquad \mathcal{S} = \frac{S}{\sqrt{\lambda}} \ ,$$

$$\mathcal{E}(\mathcal{S})_{\mathcal{S}\gg 1} = \mathcal{S} + a_0 \ln \mathcal{S} + a_c + \frac{1}{\mathcal{S}}(a_{11} \ln \mathcal{S} + a_{10})$$
$$+ \frac{1}{\mathcal{S}^2}(a_{22} \ln^2 \mathcal{S} + a_{21} \ln \mathcal{S} + a_{20}) + \mathcal{O}(\frac{\ln^3 \mathcal{S}}{\mathcal{S}^3})$$

$$a_0 = \frac{1}{\pi}, \quad a_c = \frac{1}{\pi}(\ln 8\pi - 1)$$

Expect the same structure when including string loop corrections – check at 1-loop. Structure is:

$$E = S + f \ln S + f_c + \frac{1}{S}[f_{11} \ln S + f_{10}]$$
$$+ \frac{1}{S^2}[f_{22} \ln^2 S + f_{21} \ln S + f_{20}] + \mathcal{O}(\frac{\ln^3 S}{S^3})$$

coefficients f, f_c, f_11, … receive $\frac{1}{(\sqrt{\lambda})^n}$ corrections:

$$f = \frac{\sqrt{\lambda}}{\pi}\left(1 - \frac{3\ln 2}{\sqrt{\lambda}} + \ldots\right) \qquad f_{11} = \frac{\lambda}{2\pi^2}\left(1 - \frac{6\ln 2}{\sqrt{\lambda}} + \ldots\right)$$

$$f_c = \frac{\sqrt{\lambda}}{\pi}\left(\ln\frac{8\pi}{\sqrt{\lambda}} - 1 - \frac{3\ln 2}{\sqrt{\lambda}}\ln\frac{8\pi}{\sqrt{\lambda}} + \ldots\right)$$

$$f_{10} = \frac{\lambda}{2\pi^2}\left[\ln\frac{8\pi}{\sqrt{\lambda}} - 1 - \frac{3\ln 2}{\sqrt{\lambda}}\left(2\ln\frac{8\pi}{\sqrt{\lambda}} - 1\right) + \ldots\right]$$

String side: $\sqrt{\lambda} \gg 1$ $\frac{S}{\sqrt{\lambda}}$ =fixed and then $\frac{S}{\sqrt{\lambda}} \gg 1$

Gauge theory side: $\lambda \ll 1$, S=fixed and then $S \gg 1$

Remarkable one obtains the same structure

$$\gamma(S)_{S\gg 1} = f\,\ln\bar{S} + \bar{f}_c + \frac{f_{11}\,\ln\bar{S} + \bar{f}_{10}}{S} + \frac{f_{22}\,\ln^2\,\bar{S} + \bar{f}_{21}\,\ln bS + \bar{f}_{20}}{S^2} + \\ + \frac{f_{33}\,\ln^3\,\bar{S} + \bar{f}_{32}\,\ln^2\,\bar{S} + \bar{f}_{31}\,\ln\bar{S} + \bar{f}_{30}}{S^3} + \mathcal{O}(\frac{\ln^4\,bS}{S^4}).$$

$\bar{S} = e^{\gamma_E} S$ Coefficients are power series in $\hat{\lambda} = \frac{\lambda}{16\pi^2}$

Functions f, f_c, f_11,… are interpolating functions

Anomalous dimension for twist two scalar operators $\mathrm{Tr}(\Phi D_+^S \Phi)$ at four loops obtained from asymptotic BA. (Kotikov, Lipatov, Rej, Staudacher, Velizhanin, 07)

$$f = 8\hat{\lambda} - \frac{8\pi^2}{3}\hat{\lambda}^2 + \frac{88\pi^4}{45}\hat{\lambda}^3 - (\frac{584\pi^6}{315} + 64\zeta_3^2)\hat{\lambda}^4$$

$$\bar{f}_c = -24\zeta_3\hat{\lambda}^2 + (\frac{16}{3}\pi^2\zeta_3 + 160\zeta_5)\hat{\lambda}^3 + (-\frac{56}{15}\pi^4\zeta_3 - \frac{80}{3}\pi^2\zeta_5 - 1400\zeta_7)\hat{\lambda}^4$$

$$f_{11} = 32\hat{\lambda}^2 - \frac{64\,\pi^2}{3}\hat{\lambda}^3 + \frac{96\,\pi^4}{5}\hat{\lambda}^4$$

f is universal function related to cusp anomaly of light-like Wilson loops

Interesting property: coefficients of $\frac{\ln^k S}{S^k}$ seem to be universal in twist and flavor.

all these coefficients can be determined from f:

$$\gamma(S)_{S\gg 1}$$

$$= f \ln S + f_c + \frac{f_{11} \ln S + f_{10}}{S} + \frac{f_{22} \ln^2 S + f_{21} \ln S + f_{20}}{S^2}$$

$$+ \frac{f_{33} \ln^3 S + f_{32} \ln^2 S + f_{31} \ln S + f_{30}}{S^3} + \mathcal{O}(\frac{\ln^4 S}{S^4})$$

$$f_{11} = \frac{1}{2} f^2, \qquad f_{22} = -\frac{1}{8} f^3, \qquad f_{33} = \frac{1}{24} f^4, \quad \cdots$$

Why these functional relations happen?
(B. Basso, G.P. Korchemsky, 07)

-- operators $\mathrm{tr}(\mathrm{D}_+^S \Phi^J)$ classified according to representations of SL(2,R) subgroup of SO(2,4)

-- representations labeled by conformal spin $s = \frac{1}{2}(S + \Delta)$

-- argue that anomalous dimension is a function of S only through conformal spin

$$\Delta = S + J + \gamma(S, J)$$

--implies the existence of a simpler function f

$$\boxed{\gamma(S) = \mathrm{f}\left(S + \tfrac{1}{2}\gamma(S)\right)}$$

``functional relation''

Function f simpler and more fundamental:
should not contain $\frac{\ln^k S}{S^k}$ in large S

gauge theory large S expansion consistent with functional relation:

$$\gamma(S) = f \ln\left(S + \tfrac{1}{2} f \ln S + \ldots\right) + \ldots$$

$$= f \ln S + \frac{f^2}{2} \frac{\ln S}{S} - \frac{f^3}{8} \frac{\ln^2 S}{S^2} + \frac{f^4}{24} \frac{\ln^3 S}{S^3} + \ldots$$

This gives f_{11} , f_{22} , f_{33} , ….. in terms of f

Indeed consistent with gauge theory perturbative expansions

Another interesting observed fact: reciprocity property

Function f in functional relation at large S runs in inverse even powers of quadratic Casimir of SL(2,R)

$$\mathrm{f}(S) = \sum_{n=0}^{\infty} \frac{a_n(\ln C)}{C^{2n}}$$

C is bare quadratic operator defined in terms of conformal spin $C^2 \equiv s_0(s_0 - 1)$ or in terms of spins

$$C^2 = (S + \tfrac{1}{2}J)(S + \tfrac{1}{2}J - 1)$$

Reciprocity condition implies relations among some of the coefficients of $\frac{\ln^k S}{S^m},\ k < m$

For twist $J = 2$

$$f_{10} = \frac{1}{2} f (f_c + 1)$$

$$f_{32} = \frac{1}{16} f \left[f^3 - 2 f^2 (f_c + 1) - 16 f_{21} \right]$$

Functional relation and reciprocity hold at strong coupling ? Yes, check to 1-loop in string theory

Functions f , f_c , f_{10} , f_{11} extended at strong coupling

1-loop correction at strong coupling -- some details

Expand in large semi-classical parameter $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$

Expanding in small η quadratic fluctuation Lagrangian

$$\tilde{L}_B = \tilde{L}_0 + \eta \tilde{L}_1 + \ldots$$

$$\tilde{L}_0 = - \quad \partial_a \chi \partial^a \chi + \partial_a \xi \partial^a \xi + 2\kappa_0 \ \chi \xi' - \ 2\kappa_0 \ \chi' \xi - \ 4\kappa_0 \ \tilde{\rho} \dot{\xi}$$

$$+ \quad \partial_a \tilde{\rho} \partial^a \tilde{\rho} + \partial_a \beta_u \partial^a \beta_u + 2\kappa_0^2 \beta_u^2 + \partial_a \varphi \partial^a \varphi + \partial_a \chi_s \partial^a \chi_s$$

$$\tilde{L}_1 = -\kappa_0^2 \cosh(2\kappa_0 \sigma) \xi^2 - \kappa_0^2 \cosh(2\kappa_0 \sigma) \tilde{\rho}^2 - \kappa_0^2 \sinh(2\kappa_0 \sigma) \ \xi \chi$$

$$- \frac{\kappa_0}{\pi} [\kappa_0 \pi \cosh(2\kappa_0 \sigma) - 2] \beta_u^2 + \quad (\chi \xi' - \xi \chi') [\frac{1}{\pi} - \frac{\kappa_0}{2} \cosh(2\kappa_0 \sigma)]$$

$$- \tilde{\rho} \dot{\chi} \kappa_0 \sinh(2\kappa_0 \sigma) - \tilde{\rho} \dot{\xi} [\frac{2}{\pi} + \kappa_0 \cosh(2\kappa_0 \sigma)]$$

1-loop effective action Γ_1

$$= - \frac{\mathcal{T}}{4\pi} \int_{-\infty}^{\infty} d\omega \Bigg[8 \ln \frac{\det[-\partial_1^2 + \omega^2 + \rho'^2]}{\det[-\partial_1^2 + \omega^2 + \kappa_0^2]} - 2 \ln \frac{\det[-\partial_1^2 + \omega^2 + 2\rho'^2]}{\det[-\partial_1^2 + \omega^2 + 2\kappa_0^2]}$$

$$+ \ln \frac{\det^8[-\partial_1^2 + \omega^2 + \kappa_0^2]}{\det^2[-\partial_1^2 + \omega^2 + 2\kappa_0^2] \det^6[-\partial_1^2 + \omega^2]} - \ln \frac{\det Q_\omega}{\det Q_\omega^{(0)}} + \ln \frac{\det P_\omega}{\det Q_\omega^{(0)}} \Bigg]$$

Expand ratio of determinants with

$$\ln \frac{\det[A + \eta B]}{\det A} = \eta \, \mathrm{Tr}[A^{-1} B] + \mathcal{O}(\eta^2)$$

Obtain a contribution

$$\Gamma_1^{(1)} = - \frac{\mathcal{T}\eta}{4\pi} \sum_{n=-\infty}^{\infty} A_n$$

$$A_n = \frac{8\kappa_0}{\sqrt{n^2+\kappa_0^2}} - \frac{4\kappa_0}{\sqrt{n^2+2\kappa_0^2}} - \frac{4\kappa_0}{\sqrt{n^2+4\kappa_0^2}}$$

another contribution

$$E_1^{(0)} = \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} \left[2\sqrt{n^2+2\kappa_0^2} + \sqrt{n^2+4\kappa_0^2} + 5\sqrt{n^2} - 8\sqrt{n^2+\kappa_0^2} \right]$$

Extract leading order at large using Euler-MacLaurin formula

$$\kappa_0 \equiv \frac{1}{\pi} \ln \frac{16}{\eta}$$

1-loop the same structure as at classical level

$$E_1 = b_0 \ln \mathcal{S} + b_c + \frac{b_{11} \ln \mathcal{S} + b_{10}}{\mathcal{S}} + \mathcal{O}(\frac{\ln^2 \mathcal{S}}{\mathcal{S}^2})$$

$$b_0 = -\frac{3\ln 2}{\pi} \qquad b_c = -\frac{3\ln 2}{\pi}\ln 8\pi$$

$$b_{11} = -\frac{3\ln 2}{\pi^2} \qquad b_{10} = -\frac{3\ln 2}{\pi^2}\left(\ln 8\pi - \frac{1}{2}\right)$$

functional and reciprocity relations at strong coupling imply:

$$b_{11} = a_0 b_0 \qquad b_{10} = \tfrac{1}{2}(a_0 b_c + b_0 a_c)$$

recalling classical values

$$a_0 = \frac{1}{\pi} \qquad a_c = \frac{1}{\pi}(\ln 8\pi - 1)$$

satisfied by the above coefficients !

Conclusions

- developed method to compute 1-loop corrections to spinning folded string in particular limits: long and short spinning string
- Long string: relations among coefficients of energy expansion in large S shown to hold at strong coupling to a few orders log S, S^0, 1/S, log S/S

  interesting: check this at higher orders in large S expansion. Also, extend to (S,J) solution.

  interesting: understand better functional and reciprocity relations on both gauge and string theory
- Short string: structure of energy expansion obtained to 1-loop at strong coupling

  interesting: understand BA for short operators S~1